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Approximation of truncated Beta operator of Max-product kind

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Abstract.

In this paper, we studied the Shepard nonlinear operator of max-product Beta operators. Also, we estimate the order of uniform approximation for the function $f \in C[0,1]$ to study the truncated of Beta operator of max-product. We proves that the order of uniform approximation in the general case of this kind of the approximation is $\omega_1(f,.)$ cannot be improved.

Key words and phrases, Nonlinear truncated Beta operator of max-product kind, degree of approximation, shape preserving properties.

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1.Introduction

Recently, many studies deal with the max-product and the Shepard nonlinear operator for many sequences of operators were starting the open problem 5.5.4, pp.324-326 in [4]. This branch began with Bede et al. in 2006 and 2008 when they studied the Shepard nonlinear operators of max-product kinds. After that, in 2010, Barnabas et al. introduced the max-product type operators for Favard-Szãsz-Mirakjan. In 2011 Barnabas et al. [2] and [3] were studied the nonlinear max-product Baskakov operators.

In this paper, we study the Shepard nonlinear operator of max-product Beta operators of max-product. We define these operators as follows:

$$R_{n,M}(f)(x) = \frac{\bigvee_{k=0}^{\infty} \beta_{n,k}(x) f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^{\infty} \beta_n(x)}, n \in N = \{1,2,\dots\}$$

Note that the notation $\bigvee$ denotes the maximum, where

$$\beta_{n,k}(x) = \frac{(n+k)!}{k!(n-1)!} x^k (1+k)^{-n-k-1}, \qquad n \in N, x \in [0,1].$$

Also, we introduce the order of uniform approximation for the function $f \in C[0,1]$ to study the truncated of Beta operators of max-product, which is defined as follows:

$$\mathcal{B}_{n,M}(f)(x) = \frac{\bigvee_{k=0}^{n} \beta_{n,k}(x) f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^{n} \beta_{n,k}(x)}$$

$x \in [0,1]$ and $n \in N, n \geq 1$. Then in general case of max-product of the approximation we prove that the order approximation is $\omega_1(f,.)$ cannot be improved.

Let $\mathbb{R}^+$ denotes the set of all positive real numbers, the operation $\bigvee$ is maximum and . is the product. Hence, we have $(\mathbb{R}^+, \bigvee, .)$ is a semiring structure and said to be a max-product algebra(by the operation on semiring structure and semiring properties)[1].

Let I be a closed (bounded or unbounded) interval and $CB_+(I) = \{f: I \to \mathbb{R}^+, f$ continuous and bounded on $I\}$. Then we obtain the general form $L_n: CB_+(I) \to CB_+(I)$ and (L_n said to be the discrete max-product type approximation operator) [1].

$$L_n(f)(x) = \bigvee_{k=0}^{n} k_n(x, x_i) f(x_i)$$

and

$$L_n(f)(x) = \bigvee_{k=0}^{\infty} k_n(x, x_i) f(x_i),$$

where $n \in N, f, k_n(.,k_i) \in CB_+(I)$ and $x_i \in I$ for any i.

These operators are nonlinear, positive operators and further that satisfy a pseudo linearity condition,

$$L_n(\delta.f \bigvee \beta.g) = \delta.L_n(f)(x) \bigvee \beta.L_n(g)(x),$$

for all $\delta, \beta \in \mathbb{R}^+, f, g: I \to \mathbb{R}$ [1].

First, we give the following Lemma which shows some properties of the operators L_n.

Lemma 1.1.[1]

Let $I \in \mathbb{R}$ be a bounded or unbounded interval,

$$CB_+(I) = \{ f: I \to \mathbb{R}, f \text{ continuous and bounded on } I \},$$

and $L_n : CB_+(I) \to CB_+(I), n \in N$, be a sequence of operators satisfying the following properties:

(i) If $f, g \in CB_+(I)$ satisfy $f \leq g$ then $L_n(f) \leq L_n(g)$ for all $n \in N$;

$(ii) L_n(f+g) \leq L_n(f) + L_n(g)$ for all $f, g \in CB_+(I)$.

Then for all $f, g \in CB_+(I), n \in N$ and $x \in I$ we have

$$|L_n(f)(x) - L_n(g)(x)| \leq L_n(|f - g|)(x).$$

Remark 2.1. [1]

1. One can check that the truncated Beta max-product operator satisfy the condition of Lemma 2.1, $(i), (ii)$, and satisfies the stronger condition is

$$L_n(f \vee g)(x) = L_n(f)(x) \vee L_n(g)(x), \quad f, g \in CB_+(I).$$

Indeed, we take in the above equality $f \leq g,\ f, g \in CB_+(I)$. It easily follows:

$$L_n(f)(x) \leq L_n(g)(x).$$

2. In addition, we see that the truncated Beta max-product operator is positive homogenous,

$$i.e. \qquad L_n(\rho f) = \rho L_n(f)\,, \quad \forall\, \rho \geq 0\,.$$

3. Since in the main results take $I = [0,1]$ the following two Corollaries are stated here just that the interval I is bounded and not unbounded.

Corollary 1.1. [1]

Let I be bounded or unbounded interval, $L_n: CB_+(I) \to CB_+(I)$, $n \in N$, be a sequence of operators satisfying the condition $(i), (ii)$ in Lemma 2.1, and in addition begin positively homogenous. Then for all $f \in CB_+(I), n \in N$, and $x \in I$ we have:

$$|f(x) - L_n(f)(x)| \leq [\frac{1}{\delta} L_n(\psi_x)(x) + L_n(e_0)(x)]\omega_1(f;\delta)_I + |f(x)|.|L_n(e_0)(x) - 1|,$$

where $\omega_1(f;\delta)_I = \max\{|f(x) - f(y)|; x, y \in I, |x - y| \leq \delta\}$.

An immediate consequence of Corollary 1.1. is the following:

Corollary 1.2. [1]

Suppose that in addition to the condition in the Corollary 1.1, the sequence $(L_n)_n$ satisfies $L_n(e_0) = e_0$, for all $n \in N$. Then for all $f \in CB_+(I),\ n \in N$ and $x \in I$ we have

$$|f(x) - L_n(f)(x)| \leq [1 + \tfrac{1}{\delta} L_n(\psi_x)(x)]\omega_1(f;\delta)_I.$$

2. Auxiliary Result

We state some auxiliary results which are help as in proving the main results.

Remark 2.2.

From the bounded interval which is $I = [0,1]$, note that the consideration, is that $\mathcal{B}_{nM}(f)(x)$ satisfy the conditions in Lemma 1.1. , Corollary 1.1. and Corollary 1.2.

Lemma 2.1.

The operator $\mathcal{B}_{n,M}(f)(x)$ is positive, bounded, continuous on $[0,1]$, for any arbitrary function $f: [0,1] \to \mathbb{R}^+$and satisfies $\mathcal{B}_{n,M}(f)(0) = f(0)$ for all $n \in N$.

Proof .

From, the operator $\mathcal{B}_{n,M}(f)(x)$ coincides with $f(x)$ at $x = 0$ directly follows from the consideration which is $\beta_{n,k}(x) > 0$ for all $x \in (0,1]$, $n \in N, k \in \{0,1,2, \dots, n\}$, it follows that the denominator $\bigvee_{k=0}^{n} \beta_{n,k}(x) > 0$ for all $x \in (0,1]$, and $n \in N$.

But the numerator is a maximum of finite number of continuous function on $[0,1]$, so it is a continuous function on $[0,1]$ and this implies that $\mathcal{B}_{n,M}(f)(x)$ is continuous on $(0,1]$.

To prove that the continuity of $\mathcal{B}_{n,M}(f)(x)$ at $x=0$, we observe that $\beta_{n,k}(x)=0$, for all $k \in \{0,1,2,\ldots,n\}$ and $\beta_{n,k}(x)=1$ for $k=0$ which implies that $\bigvee_{k=0}^{n} \beta_{n,k}(x)=1$ in the case of $x=0$.

The operator $\mathcal{B}_{n,M}(f)(x)$ coincides with $f(x)$ at $x=0$ directly follows from the above consideration .

Which proves the Lemma .

□

Remark 2.3.

In view of Lemma 2.1, we have $\mathcal{B}_{n,M}(f)(0)=f(0)$ for all n, through it, follow that in the notations, proofs and statement of all approximation results, in fact we always may suppose that $x>0$. For each $n \in N,\ n \geq 1, k \in \{0,1,2,\ldots,n\}, j \in \{0,1,2,\ldots,n-1\}$ and $x \in \left[\frac{j}{n},\frac{j+1}{n}\right]$, let us denote $M_{k,n,j}(x)=m_{k,n,j}(x) \mid \frac{k}{n}-x \mid$,

where $m_{k,n,j}(x)=\frac{\beta_{n,k}(x)}{\beta_{n,j}(x)}$, for all $x \in (0,1]$ and $k \in \{1,2,\ldots,n\}$. Clearly that $m_{0,n,0}(0)=1, \forall\ x \in (0,1]$, by the same way, we have: $m_{k,n,0}(0)=0$. For the Lemma 2.1,

if $k>j$ then $\quad M_{k,n,j}(x)=m_{k,n,j}(x)\left(\frac{k}{n}-x\right)$

and if $k \leq j$ then $\quad M_{k,n,j}(x)=m_{k,n,j}(x)\left(x-\frac{k}{n}\right)$

To prove the mains results, we need to introduce some auxiliary results, such as:

Lemma 2.2.

Let $n \in N$. For all $k \in \{0,1,2,\ldots,n\}, j \in \{0,1,2,\ldots,n-1\}$ and $x \in \left[\frac{j}{n},\frac{j+1}{n}\right]$ we have

$$m_{k,n,j}(x) \leq 1.$$

Proof .

First let $x=0$ we have $j=0$ it is clearly $m_{0,n,0}(x)=1$, by the same way we have: $m_{k,n,0}(0)=0$ for all $k \in \{0,1,2,\ldots,n\}$, Suppose that $x>0$ then, it is clearly that $m_{k,n,j}(x)>0$, there is two cases: 1)$k \geq j$ and 2) $k \leq j$.

Case 1). Since the function $h(x)=\frac{1+x}{x}$, it is nonincreasing on $\left[\frac{j}{n},\frac{j+1}{n}\right]$ and (when $j=0$ then $\left(0,\frac{1}{n}\right]$) we get:
$$\frac{m_{k,n,j}(x)}{m_{k+1,n,j}}=\frac{(n+k)!\,x^{k}(1+x)^{-n-k-1}}{k!\,(n-1)!}\cdot\frac{(k+1)!\,(n-1)!}{(n+k+1)!\,x^{k+1}(1+x)^{-n-k-2}}$$
$$=\frac{k+1}{n+k+1}\cdot\frac{1+x}{x}. \qquad \text{which implies that}$$
$$\frac{m_{k,n,j}(x)}{m_{k+1,n,j}} \geq \frac{k+1}{n+k+1}\cdot\frac{1+\frac{j+1}{n}}{\frac{j+1}{n}}=\frac{(k+1)(n+j+1)}{(n+k+1)(j+1)} \geq 1. \text{ which implies that}$$
$$m_{j,n,j}(x) \geq m_{j+1,n,j}(x) \geq m_{j+2,n,j}(x) \geq \cdots \geq m_{n,n,j}(x).$$

Case 2) since
$$\frac{M_{k,n,j}(x)}{M_{k-1,n,j}(x)}=\frac{(n+k)!\,x^{k}(1+x)^{-n-k-1}}{k!\,(n-1)!}\cdot\frac{(k-1)!\,(n-1)!}{(n+k-1)!\,x^{k-1}(1+x)^{-n-k}}\frac{k+1}{k}\cdot\frac{x}{1+x}\ .$$

we obtain $\frac{m_{k,n,j}(x)}{m_{k-1,n,j}} \geq \frac{k+1}{k} \cdot \frac{\frac{j}{n}}{1+\frac{j}{n}} = \frac{j(k+1)}{k(n+j)} \geq 1$, which implies that

$$m_{j,n,j}(x) \geq m_{j-1,n,j}(x) \geq m_{j-2,n,j}(x) \geq \cdots \geq m_{0,n,j}(x).$$

Since $m_{j,n,j}(x) = 1$ and this proves of the Lemma. □

Lemma 2.3.

Let $x \in \left[\frac{j}{n}, \frac{j+1}{n}\right]$ and $n \in N$.

(i) If $k \in \{j+3, j+4, \dots, n-1\}$ is such that $k - \sqrt{2(k+1)} \geq j$, then $M_{k,n,j}(x) \geq M_{k+1,n,j}(x)$.

(ii) If $k \in \{1,2,\dots,j-1\}$ is such that $j - \sqrt{2j} \geq k$, then $M_{k,n,j}(x) \geq M_{k-1,n,j}(x)$.

Proof.

(i) We note that

$$\frac{M_{k,n,j}(x)}{M_{k+1,n,j}(x)} = \frac{m_{k,n,j}(x)\left(\frac{k}{n} - x\right)}{m_{k+1,n,j}(x)\left(\frac{k+1}{n} - x\right)}$$

$$= \frac{(n+k)!\, x^k (1+x)^{-n-k-1}}{k!\,(n-1)!} \cdot \frac{(k+1)!\,(n-1)!}{(n+k+1)!\, x^{k+1}(1+x)^{-n-k-2}} \cdot \frac{\frac{k}{n} - x}{\frac{k+1}{n} - x}$$

$$= \frac{k+1}{n+k+1} \cdot \frac{1+x}{x} \cdot \frac{\frac{k}{n} - x}{\frac{k+1}{n} - x}.\ \textit{Since the function } g(x) = \frac{1+x}{x} \cdot \frac{\frac{k}{n} - x}{\frac{k+1}{n} - x}$$

is nondecreasing , it follows that

$$g(x) \leq g\left(\frac{j+1}{n}\right) = \frac{1+\frac{j+1}{n}}{\frac{j+1}{n}} \cdot \frac{\frac{k}{n} - \frac{j+1}{n}}{\frac{k+1}{n} - \frac{j+1}{n}} = \frac{n+j+1}{j+1} \cdot \frac{k-j-1}{k-j},$$

for all $x \in \left[\frac{j}{n}, \frac{j+1}{n}\right]$. Then $M_{k+1,n,j}(x) \leq M_{k,n,j}(x) \frac{k+1}{n+k+1} \cdot \frac{n+j+1}{j+1} \cdot \frac{k-j-1}{k-j}$

Since the condition $k - \sqrt{2(k+1)} \geq j$, and using calculation

We obtain $(k+1)(n+j+1)(k-j-1) - (n+k+1)(j+1)(k-j)$

$$= n[(k-j)^2 - (k+1)] - (k+1)(j+1).$$

(ii) we note that

$$\frac{M_{k,n,j}(x)}{M_{k-1,n,j}(x)} = \frac{(n+k)!\, x^k (1+x)^{-n-k-1}}{k!\,(n-1)!} \cdot \frac{(k-1)!\,(n-1)!}{(n+k-1)!\, x^{k-1}(1+x)^{-n-k}} \cdot \frac{x - \frac{k}{n}}{x - \frac{k-1}{n}}$$

$= \frac{n+k}{k} \cdot \frac{x}{1+x} \cdot \frac{x - \frac{k}{n}}{x - \frac{k-1}{n}}$. since the function $h(x) = \frac{x}{1+x} \cdot \frac{x - \frac{k}{n}}{x - \frac{k-1}{n}}$ is nondecreasing,

it follows that $h(x) \geq h\left(\frac{j}{n}\right) = \frac{\frac{j}{n}}{1+\frac{j}{n}} \cdot \frac{\frac{j}{n} - \frac{k}{n}}{\frac{j}{n} - \frac{k-1}{n}} = \frac{j}{n+j} \cdot \frac{j-k}{j-k+1}$

for all $x \in \left[\frac{j}{n}, \frac{j+1}{n}\right]$. Then $M_{k,n,j}(x) \geq M_{k-1,n,j}(x)\, \frac{n+k}{k} \cdot \frac{j}{n+j} \cdot \frac{j-k}{j-k+1}$.
Since the condition $j - \sqrt{2j} \geq k$, and using calculation we obtain
$(n+k)j(j-k) - k(n+j)(j-k+1) = n[(j-k)^2 - k] - kj \geq 0$ which prove Lemma □

Lemma 2.4.

$Let\ n \in N, we\ have\ \ \bigvee_{k=0}^{n} \beta_{n,k}(x) = \beta_{n,j}(x)$, for all $x \in \left[\frac{j}{n}, \frac{j+1}{n}\right]$, $j \in \{0,1,2,\dots,n-1\}$.

Proof .

First we show that for fixed $n \in N$ and $0 \leq k \leq k+1 \leq n$ we have
$0 \leq \beta_{n,k+1}(x) \leq \beta_{n,k}(x)$ if and only if $x \in \left[0, \frac{k+1}{n}\right]$.

$$0 \leq \frac{(n+k+1)!}{(k+1)!(n-1)!}\, x^{k+1}(1+x)^{-n-k-2} \leq \frac{(n+k)!}{k!(n-1)!}\, x^{k}(1+x)^{-n-k-1}.$$

Which after simple calculus is obviously equivalent to $0 \leq x \leq \frac{k+1}{n}$,
now, by taking $k = 0,1,2,\dots,n-1$, in the inequality just prove above , we obtain

$\beta_{n,1}(x) \leq \beta_{n,0}(x)$ if and only if $x \in \left[0, \frac{1}{n}\right]$,

$\beta_{n,2}(x) \leq \beta_{n,1}(x)$ if and only if $x \in \left[0, \frac{2}{n}\right]$,

$\beta_{n,3}(x) \leq \beta_{n,2}(x)$ if and only if $x \in \left[0, \frac{3}{n}\right]$,

so on, $\beta_{n,k+1}(x) \leq \beta_{n,k}(x)$ if and only if $x \in \left[0, \frac{k+1}{n}\right]$, and so on until finally,

$\beta_{n,n-1}(x) \leq \beta_{n,n-2}(x)$ if and only if $x \in [0,1]$ and

$\beta_{n,n}(x) \leq \beta_{n,n-1}(x)$ if and only if $x \in [0,1]$.

From all these inequalities , reasoning by recurrence we easily obtain ,

if $x \in \left[0, \frac{1}{n}\right]$ then $\beta_{n,k}(x) \leq \beta_{n,0}(x)$ for all $k = 0,1,2,\dots,n$,

if $x \in \left[\frac{1}{n}, \frac{2}{n}\right]$ then $\beta_{n,k}(x) \leq \beta_{n,1}(x)$ for all $k = 0,1,2\dots,n$,

if $x \in \left[\frac{2}{n}, \frac{3}{n}\right]$ then $\beta_{n,k}(x) \leq \beta_{n,2}(x)$ for all $k = 0,1,2,\dots n$,

and in the case general then if $x \in \left[\frac{j}{n}, \frac{j+1}{n}\right]$ then $\beta_{n,k}(x) \leq \beta_{n,j}(x)$ for all $k = 0,1,2,\dots n$.
From the application above we can write the equation as follows :
$\bigvee_{k=0}^{n} \beta_{n,k}(x) = \max\{\bigvee_{k=0}^{j-1} \beta_{n,k}(x), \bigvee_{k=j}^{n} \beta_{n,k}(x)\}$, which prove of Lemma □ .

Suppose that, for any $n \in N, k \in \{0,1,2,\dots,n\}$ and $j \in \{0,1,2,\dots,n-1\}$, let us define the functions $f_{k,n,j} : \left[\frac{j}{n}, \frac{j+1}{n}\right] \to \mathbb{R}$,

$$f_{k,n,j}(x) = m_{k,n,.j}(x) f\left(\frac{k}{n}\right) = \frac{\beta_{n,k}(x)}{\beta_{n,j}(x)}\, f\left(\frac{k}{n}\right) = \frac{(n+k)!}{(n+j)!} \cdot \frac{j!}{k!} \left(\frac{x}{1+x}\right)^{k-j} f\left(\frac{k}{n}\right).$$

For any $j \in \{0,1,2,\dots,n-1\}$ and $x \in \left[\frac{j}{n}, \frac{j}{n}\right]$ we can write $\mathcal{B}_{n,M}(f)(x) = \bigvee_{k=0}^{n} f_{k,n,j}(x)$.

4. Approximation Results

Statement not clear, if $\mathcal{B}_{n,M}(f)(x)$ represents the truncated Beta operator of max-product kind, then the first main result of this section is the following .

Lemma 2.5.

Let $f : [0,1] \to [0,\infty)$ be such that $\mathcal{B}_{n,m}(f)(x) = \max\{f_{j,n,j}(x), f_{j+1,n,j}(x)\}$ for all $x \in \left[\frac{j}{n}, \frac{j+1}{n}\right]$ and $n \in N$. Then $|\mathcal{B}_{n,M}(f)(x) - f(x)| \le \omega_1\left(f; \frac{1}{n}\right)$ for all $x \in \left[\frac{j}{n}, \frac{j+1}{n}\right]$.

Proof .

We take two cases : ***Case 1).*** Let $x \in \left[\frac{j}{n}, \frac{j+1}{n}\right]$ is fixed such that $\mathcal{B}_{n,M}(f)(x) = f_{k,n,j}(x)$. By simple calculation we have $0 \le x - \frac{j}{n} \le \frac{j+1}{n} - \frac{j}{n} = \frac{1}{n} \le \frac{1}{n}$, and $f_{j,n,j}(x) = f\left(\frac{j}{n}\right)$, it follows that $|\mathcal{B}_{n,M}(f)(x) - f(x)| \le \omega_1\left(f; \frac{1}{n}\right)$.

Case 2). Let $x \in \left[\frac{j}{n}, \frac{j+1}{n}\right]$ be such that $\mathcal{B}_{n,M}(f)(x) = f_{j+1,n,j}(x)$. We have two subcases:

a) $\mathcal{B}_{n,M}(f)(x) \le f(x)$, when obviously, $f_{j,n,j}(x) \le f_{j+1,n,j}(x) \le f(x)$ and we get

$$|\mathcal{B}_{n,M}(f)(x) - f(x)| = |f_{j+1,n,j}(x) - f(x)| = f(x) - f_{j+1,n,j}(x) \le f(x) - f\left(\frac{j}{n}\right)$$
$$\le \omega_1\left(f; \frac{1}{n}\right)$$

b) $\mathcal{B}_{n,M}(f)(x) > f(x)$, when $|\mathcal{B}_{n,M}(f)(x) - f(x)| = f_{j+1,n,j} = m_{j+1,n,j}(x) f\left(\frac{j+1}{n}\right) - f(x) \le f\left(\frac{j+1}{n}\right) - f(x)$. Since $0 \le \frac{j+1}{n} - x = \frac{j+1}{n} - \frac{j}{n} = \frac{1}{n} \le \frac{1}{n}$, it follows that

$$f\left(\frac{j+1}{n}\right) - f(x) \le \omega_1\left(f; \frac{1}{n}\right)$$ □

Theorem 4.1.

Let $f : [0,1] \to \mathbb{R}_+$ be continuous . Then we have the estimate

$$|\mathcal{B}_{n,M}(f)(x) - f(x)| \le 24\omega_1\left(f; \frac{1}{\sqrt{n+1}}\right), n \in N, \ x \in [0,1], \text{ where}$$
$$\omega_1(f,\delta) = \sup\{|f(x) - f(y)|; x, y \in [0,1], |x - y| \le \delta\}.$$

Proof.

One check that the truncated max-product Beta operator execute the condition Corollary 2.3 and we have

$$|\mathcal{B}(f)(x) - f(x)| \le \left(1 + \frac{1}{\delta_n}\mathcal{B}_{n,M}(\psi_x)(x)\right)\omega_1(f; \delta_n), \quad \ldots.. \quad (1)$$

where $\psi_x(t) = |t - x|$. So, it is enough to estimate

$$E_n(x) \coloneqq \mathcal{B}_{n,M}(\psi_x)(x) = \frac{\bigvee_{k=0}^{n} \beta_{n,k}(x) \left|\frac{k}{n} - x\right|}{\bigvee_{k=0}^{n} \beta_{n,k}(x)}, x \in [0,1]. \text{ Let } x \in \left[\frac{j}{n}, \frac{j+1}{n}\right], \text{where}$$

$j = \{0,1,\ldots,n-1\}$ is fixed . By Lemma 2.4, we obtain

$$E_n(x) = \max_{k=0,1,2,\ldots} \{M_{k,n,j}(x)\}, \quad x \in \left[\frac{j}{n}, \frac{j+1}{n}\right],$$

we suppose that $j \in \{0,1,2,\ldots,n-1\}$ because for $j = 0$ using evaluation when $j = 0$ which implies that, since $x \in \left[\frac{j}{n}, \frac{j+1}{n}\right]$ then $x \in \left[\frac{0}{n}, \frac{0+1}{n}\right]$ implies that $x \in \left[0, \frac{1}{n}\right]$, take the value maximum of the interval $\left[0,\frac{1}{n}\right]$ shows that in this case we obtain $E_n(x) \le \frac{1}{n}$ for all $x \in \left[\frac{j}{n},\frac{j+1}{n}\right]$. Indeed, in this case we get

$$M_{k,n,j}(x) = m_{k,n,j}(x) \mid \frac{k}{n} - x \mid \frac{\beta_{n,k}(x)}{\beta_{n,j}(x)} \mid \frac{k}{n} - x \mid \ = \frac{\frac{(n+k)!}{k!\,(n-1)!}x^k(1+x)^{-n-k-1}}{\frac{(n+j)!}{j!\,(n-1)!}x^j(1+x)^{-n-j-1}} \mid \frac{k}{n} - x \mid$$

and when $j = 0$ implies that $M_{k,n,0}(x) = \dfrac{\beta_{n,k}(x)}{\dfrac{n}{(1+x)^{n+1}}}\ .\mid \dfrac{k}{n} - x \mid$ which for $k = 0$ gives,

$$M_{0,n,0}(x) \ = \frac{\frac{n!}{(n-1)!}\ (1+x)^{-n-1}}{\frac{n!}{(n-1)!}\ (1+x)^{-n-1}}\ .\mid -x \mid\ = 1\,.\,x = x.$$

Therefore, $M_{0,n,0}(x) = x\ < \dfrac{1}{n}$, in the interval $\left[0,\dfrac{1}{n}\right]$. Also, for any $k \ge 1$ we get

$$M_{k,n,0}(x)\ \le\ \frac{\beta_{n,k}(x)}{\frac{n}{(1+x)^{n+1}}}\ .\frac{k}{n}\ \le\ \frac{(n+k)!}{n!(k-1)!}\left(\frac{x}{1+x}\right)^k\ \le \frac{1}{n}\ .$$

So, it remain to get an upper estimate for each $M_{k,n,j}(x)$ when $j = \{1,2,\ldots,n-1\}$ is fixed, $x \in \left[\frac{j}{n},\frac{j+1}{n}\right]$ and $k \in \{0,1,2,\ldots,n\}$. In fact we will prove that $M_{k,n,j}(x) \le \frac{2\sqrt{3}(\sqrt{2}+2)}{\sqrt{n+1}}$, for all

$x \in \left[\frac{j}{n},\frac{j+1}{n}\right]$, $k = \{0,1,2,\ldots,n\}$ ………. (2) And which implies that

$E_n(x) \le \frac{2\sqrt{3}(\sqrt{2}+2)}{\sqrt{n+1}}$, for all $x \in [0,1]$, $n \in N$, and taking $\delta_n = \frac{2\sqrt{3}(\sqrt{2}+2)}{\sqrt{n+1}}$ in (1) ,

since $[2\sqrt{3}(\sqrt{2}+2)] = 11$,from the property which to say $\omega_1(f;\lambda\delta) \le ([\lambda]+1)\omega_1(f;\delta)$, we obtain the estimate in the statement. In order to prove (2) we take the following cases:

1) $k =$ j ; **2**) $k \ge j+1$ and ***3)*** $k \le j-1$.

Case1). If $k = j$ since

$$M_{k,n,j}(x) = \frac{\beta_{n,k}(x)}{\beta_{n,j}(x)}\ \mid \frac{k}{n} - x \mid \text{then } M_{j,n,j}(x) = \frac{\beta_{n,j}(x)}{\beta_{n,j}(x)}\ \mid \frac{j}{n} - x \mid\ =\ \mid \frac{j}{n} - x \mid\ = x - \frac{j}{n},$$

since $x \in \left[\frac{j}{n},\frac{j+1}{n}\right]$ then

$$M_{j,n,j}(x) = x - \frac{j}{n} = \frac{j+1}{n} - \frac{j}{n} = \frac{1}{n}\ .It\ follows\ that\ \ M_{j,n,j}(x) \le \frac{1}{n}$$

it follows that $\mid \frac{j+1}{n} - x \mid\ < \frac{1}{n}$ which implies that $M_{j+1,n,j}(x) < \frac{1}{n}$.

Case2.a) Suppose that $k - \sqrt{2(k+1)}\ < j$. We obtain

$$M_{k,n,j}(x) = m_{k,n,j}(x)\left(\frac{k}{n} - x\right) \le \frac{k}{n} - x \le \frac{k}{n} - \frac{j}{n} \le \frac{k}{n} - \frac{k - \sqrt{2(k+1)}}{n}$$
$$= \frac{\sqrt{2(k+1)}}{n} \le \frac{3\sqrt{2}}{\sqrt{n+1}}\,.$$

Case2.b). suppose that $k-\sqrt{2(k+1)} \geq j$. Since the function $g(x)=x-\sqrt{2(x+1)}$ is nondecreasing on the interval $[0,\infty)$ it follows that there exists $\bar{k} \in \{0,1,2,\dots n\}$, of maximum value, such that $\bar{k}-\sqrt{2(\bar{k}+1)} < j$. Then for $k_1=\bar{k}+1$ we get $k_1-\sqrt{2(k_1+1)} \geq j$. Then

$$M_{\bar{k}+1,n,j}(x)=m_{\bar{k}+1,n,j}(x)\left(\frac{\bar{k}+1}{n}-x\right) \leq \frac{\bar{k}+1}{n}-x \leq \frac{\bar{k}+1}{n}-\frac{j}{n}$$

$$\leq \frac{\bar{k}+1}{n}-\frac{\bar{k}-\sqrt{2(\bar{k}+1)}}{n}=\frac{\sqrt{2(\bar{k}+1)}+1}{n} \leq \frac{3\sqrt{2}+\sqrt{3}}{\sqrt{n+1}}.$$

Also, we have $k_1 \geq j+1$. Indeed, this is consequence of the fact the function g is nondecreasing on the interval $[0,\infty)$ and from simple calculus we obtain $g(j)<j$. By Lemma 2.3, (i) it follows that $M_{\bar{k}+1,n,j}(x) \geq M_{\bar{k}+2,n,j}(x) \geq \cdots \geq M_{n,n,j}(x)$.

We hence get on $M_{k,n,j}(x) \leq \frac{3\sqrt{2}+\sqrt{3}}{\sqrt{n+1}}$ for any $k \in \{\bar{k}+1,\bar{k}+2,\dots,n\}$.

Therefore, in both subcases, by Lemma 3.2, (i), we get $M_{k,n,j}(x) \leq \frac{3\sqrt{2}+\sqrt{3}}{\sqrt{n+1}}$.

Case3.a). In the beginning suppose that $j-\sqrt{2j}<k$. Then we obtain

$$M_{k,n,j}(x)=m_{k,n,j}(x)\left(x-\frac{k}{n}\right) \leq \frac{j+1}{n}-\frac{k}{n} \leq \frac{j+1}{n}-\frac{j-\sqrt{2j}}{n}=\frac{\sqrt{2j}+1}{n} \leq \frac{\sqrt{2}+1}{\sqrt{n+1}}.$$

b). suppose now that $j-\sqrt{2j} \geq k$. Let $\tilde{k} \in \{0,1,2,\dots,n\}$ be the minimum value such that $j-\sqrt{2j}<\tilde{k}$. Then $k_2=\tilde{k}-1$ satisfies $j-\sqrt{2j} \geq k_2$. Then

$$M_{\tilde{k}-1,n,j}(x)=m_{\tilde{k}-1,n,j}(x)\left(x-\frac{\tilde{k}-1}{n}\right) \leq \frac{j+1}{n}-\frac{\tilde{k}-1}{n} \leq \frac{j+1}{n}-\frac{j-\sqrt{2j}-1}{n}$$

$$=\frac{\sqrt{2j}+2}{n} \leq \frac{\sqrt{2}+2}{n}$$. Also, because in this case it is immediate take $k_2 \leq j-1$.

By Lemma 3.4, (ii) it follows that $M_{\tilde{k}-1,n,j}(x) \geq M_{\tilde{k}-2,n,j}(x) \geq \cdots \geq M_{0,n,j}(x)$.

We get $M_{k,n,j}(x) \leq \frac{\sqrt{2}+2}{\sqrt{n}}$ for any $k \leq j-1$ and $x \in \left[\frac{j}{n},\frac{j+1}{n}\right]$.

In both subcases, by Lemma 3.2, (ii), we get $M_{k,n,j}(x) \leq \frac{2(\sqrt{2}+2)}{\sqrt{n}} \leq \frac{2\sqrt{3}(\sqrt{2}+2)}{\sqrt{n+1}}$.

In conclusion, collection all the estimates in the above cases and subcases we obtain the relationship(2), which completes the proof. □

Remark 4.1.

The order of approximation in terms of ω_1 in Theorem 4.1 cannot improved (when $E_n(x)$ is defined in the proof of theorem 4.1 then the expression $\max_{x\in[0,1]}\{E_n(x)\}$ is exactly $\frac{1}{\sqrt{n+1}}$ Indeed, for $n \in N$ let us take $j_n=\left[\frac{n}{2}\right]$, $k_n=j_n+[\sqrt{n}]$ and $x_n=\frac{j_n+1}{n}$. Then using Calculation for all $n \geq 2$ we get

$$M_{k_n,n,j_n}(x_n)=m_{k_n,n,j_n}(x_n)\left(\frac{k_n}{n}-x\right)=\frac{\beta_{n,k_n}(x_n)}{\beta_{n,j_n}(x_n)}\left(\frac{k_n}{n}-x_n\right)$$

$$=\frac{(n+k_n)!}{(n+j_n)!}\cdot\frac{j_n!}{k_n!}\cdot\left(\frac{x_n}{1+x_n}\right)^{k_n-j_n}\cdot\left(\frac{k_n}{n}-x_n\right)$$

$$= \frac{(k_n+1)(k_n+2)\dots.j_n}{(n+k_n)(n+k_n+1)(n+k_n+2)\dots(n+j_n-1)}\cdot\left(\frac{\frac{j_n+1}{n}}{1+\frac{j_n+1}{n}}\right)^{k_n-j_n}$$

$$\cdot\left(\frac{j_n+[\sqrt{n}]}{n}-\frac{j_n+1}{n}\right).\geq\left(\frac{k_n+1}{n+j_n}\right)^{j_n-k_n}\left(\frac{\left[\frac{n}{2}\right]+1}{\left[\frac{n}{2}\right]+n+1}\right)^{k_n-j_n}\cdot\frac{1}{\sqrt{n}}$$

$$=\left(\frac{n+\left[\frac{n}{2}\right]-1}{\left[\frac{n}{2}\right]+[\sqrt{n}]+1}\right)^{[\sqrt{n}]}\cdot\left(\frac{\left[\frac{n}{2}\right]+1}{\left[\frac{n}{2}\right]+n+1}\right)^{[\sqrt{n}]}\cdot\frac{1}{\sqrt{n}}$$

$$\geq\left(\frac{\left[\frac{n}{2}\right]+[\sqrt{n}]-1}{\left[\frac{n}{2}\right]+1}\right)^{[\sqrt{n}]}\cdot\frac{1}{\sqrt{n}}\geq\left(\frac{\frac{n}{2}+\sqrt{n}}{\frac{n}{2}+1}\right)^{\sqrt{n}}\cdot\frac{1}{\sqrt{n}}\ .$$

Because $\lim_{n\to\infty}\left(\frac{\frac{n}{2}+\sqrt{n}}{\frac{n}{2}+1}\right)^{\sqrt{n}} = e^{-2}$, there exsits $n_0 \in N$ such that$\left(\frac{1+\left[\frac{n}{2}\right]}{\left[\frac{n}{2}\right]+[\sqrt{n}]}\right)^{[\sqrt{n}]} \geq e^{-3}$, for all $n \geq \max\{n_0\ ,2\}$.[1] It follows $M_{k_n,n,j_n}(x_n) \geq \frac{e^{-3}}{\sqrt{n}} \geq \frac{e^{-3}}{\sqrt{n+1}}$, for all $n \geq max\{\ n_0, 2\}$. Taking into Lemma 3.1, (ii) too, It follows that for all $n \geq \max\{n_0\ ,2\}$ we have

$M_{k_n,n,j_n}(x_n) \geq \frac{e^{-3}}{\sqrt{n+1}}$, which implies the desired conclusion .

Lemma 4.1.

Let $f : [0,1] \to [0,\infty)$ be concave. Then the function $g : (0,1] \to [0,\infty), g(x) = \frac{f(x)}{x}$ is nonincreasing.

Proof .

Let $x, y \in (0,1]$ be with $x \leq y$. Then

$$f(x) = f\left(\frac{x}{y}\ y + \frac{y-x}{y}\ 0\right) \geq \frac{x}{y}\ f(y) + \frac{y-x}{y}\ f(0) \geq \frac{x}{y}\ f(y)\ ,$$

Which implies $\frac{f(x)}{x} \geq \frac{f(y)}{y}$.

□

Lemma 4.2

Let $f : [0,1] \to [0,\infty)$ be a nondecreasing function such that the function $g : (0,1] \to [0,\infty)$,

$g(x) = \frac{f(x)}{x}$ is nonincreasing, then $|\mathcal{B}_{n,M}(f)(x) - f(x)| \leq 2\omega_1\left(f;\frac{1}{n}\right)$, for all $x \in [0,1]$, and $n \in N$.

Proof .

Since f is nondecreasing it follows that

$$\mathcal{B}_{n,M}(f)(x) = \bigvee_{k\geq j}^{n} f_{k,n,j}\ (x),\quad \text{for all}\ \ x \in \left[\frac{j}{n},\frac{j+1}{n}\right].$$

Let $x \in [0,1]$ and $j \in \{0,1,2,\dots,n-1\}$ such that $x \in \left[\frac{j}{n},\frac{j+1}{n}\right]$. Let $k \in \{0,1,\dots,n\}$ be with $k \geq j$.

Since

$$f_{k,n,j}(x) = m_{k,n,j} f\left(\frac{k}{n}\right) = \frac{\beta_{n,k}(x)}{\beta_{n,j}(x)} = \frac{(n+k)!}{(n+j)!} \cdot \frac{j!}{k!} \cdot \left(\frac{x}{1+x}\right)^{k-j} ,$$

$$f_{k+1,n,j}(x) = m_{k+1,n,j}(x) f\left(\frac{k+1}{n}\right) = \frac{\beta_{n,k+1}(x)}{\beta_{n,j}(x)} f\left(\frac{k+1}{n}\right)$$

$$= \frac{(n+k+1)!}{(n+j)!} \cdot \frac{j!}{(k+1)!} \cdot \left(\frac{x}{1+x}\right)^{k-j} \left(\frac{x}{1+x}\right) \cdot f\left(\frac{k+1}{n}\right).$$

Since $g(x)$ is nonincreasing we obtain $\frac{f\left(\frac{k+1}{n}\right)}{\frac{k+1}{n}} \leq \frac{f\left(\frac{k}{n}\right)}{\frac{k}{n}}$ that is $f\left(\frac{k+1}{n}\right) \leq \frac{k+1}{k} f\left(\frac{k}{n}\right)$. From $x \leq \frac{j+1}{n-1}$
it follows

$$f_{k+1,n,j}(x) \leq \frac{(n+k+1)!}{(n+j)!} \cdot \frac{j!}{(k+1)!} \cdot \left(\frac{x}{1+x}\right)^{k-j} \frac{j+1}{n+j} \cdot \frac{n+k}{k+1} \cdot \frac{k+1}{k} f\left(\frac{k}{n}\right)$$

$$= f_{k,n,j}(x) \frac{j+1}{n+j} \cdot \frac{n+k}{k} = \frac{(n+j)k + n(j+1-k)}{(n+j)k} \cdot f_{k,n,j}(x)$$

It is directly that for $k \geq j+2$ we have $f_{k,n,j}(x) \geq f_{k+1,n,j}(x)$. Therefore, we get

$$f_{j+2,n,j}(x) \geq f_{j+3,n,j}(x) \geq \cdots \geq f_{n,j,n}(x),$$

that is

$$\mathcal{B}_{n,M}(f)(x) = \max\{f_{j,n,j}(x), f_{j+1,n,j}(x)\},$$

for all $x \in \left[\frac{j}{n}, \frac{j+1}{n}\right]$, and from Lemma 4.1 we get

$$\left|\mathcal{B}_{n,M}(f)(x) - f(x)\right| \leq \omega_1\left(f; \frac{1}{n}\right).$$ □

Corollary 4.3.

Let $f : [0,1] \to [0,\infty)$ be a nondecreasing concave function . Then

$$\left|\mathcal{B}_{n,M}(f)(x) - f(x)\right| \leq \omega_1\left(f; \frac{1}{n}\right),$$

for all $x \in [0,1]$.

Proof .

The proof immediate by Lemma 4.3 and by Corollary 4.4. □

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تقريب البتر لمؤثر بيتا من نمط أعظم – ضرب

الخلاصة:

في هذه البحث, درسنا المؤثر اللاخطي لـ (Shepard) لمؤثر بيتا من النمط أعظم–ضرب . كذلك, قدمنا رتبة التقريب المنتظم للدالة $f \in C[0,1]$ لدراسة بتر مؤثر بيتا لأعظم – ضرب. ثم برهنا بان رتبة التقريب المنتظم في الحالة العامة لهذا النمط من التقريب يكون $\omega_1(f,.)$ لا يمكن تحسينه.

الكلمات الافتتاحية والعبارات :البتر لمؤثر بيتا اللاخطي من نمط أعظم – ضرب ودرجة التقريب والحفاظ على الخواص.